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ARBITRAGE PRICING OF EQUITY CORRELATION SWAPS

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Summary

In this report we propose a 'toy model' for pricing derivatives on the realized variance of an Asset, which we apply for pricing correlation swaps on the components of an equity index. We find that the fair strike of a correlation swap is approximately equal to a particular measure of implied correlation, and that the corresponding hedging strategy relies upon dynamic trading of variance dispersions.

I thank my colleague Manos Venardos for his contribution and comments. All errors are mine.

Introduction

Volatility and variance modeling has been an active research area within quantitative finance since the publication of the Black-Scholes model in 1973. Initially, research efforts have mostly focused on extending the Black-Scholes model for pricing calls and puts in the presence of implied volatility 'smile' (Hull-White 1987, Heston 1993, Dupire 1993a & 1993b, Derman-Kani 1994.) In the mid 1990's, new instruments known as variance swaps also appeared on equities markets and made squared volatility a tradable asset (Neuberger 1990, Demeterfi-Derman 1999.) As variance became an asset class of its own, various forms of volatility derivatives have appeared, for example volatility swaps, forward contracts and options on the new CBOE Volatility Index (VIX.) The modeling of these new instruments is difficult because they overlap with certain exotic derivatives such as cliquet options which highly depend on the dynamics of the implied volatility surface.

In recent years the research on volatility and variance modeling has embraced the pricing and hedging of these volatility derivatives. Here we must distinguish between two types:

- Derivatives on realized volatility, where the payoff explicitly involves the historical volatility of the underlying Asset observed between the start and maturity dates, e.g. volatility swaps.
- Derivatives on implied volatility, where the payoff will be determined at maturity by the implied volatility surface of the underlying asset, e.g. forward-starting variance swaps, cliquet options, or options on the VIX.

It is important to notice that the first category can be seen as derivatives on a variance swap of same maturity. Leveraging on this observation and on earlier work by Dupire (1993b), Buehler (2004) models a continuous term structure of forward variance swaps, while Duanmu (2004), Potter (2004) and Carr-Sun (2005) model a fixed-term variance swap. All these approaches are based on dynamic hedging with one or several variance swap instruments.

The second category is beyond the scope of this report. We refer the interested reader to the work on the dynamics of the implied volatility surface carried out by Schonbucher (1998), Cont-Fonseca (2002), Brace et al. (2002).

Despite the development of exotic and hybrid markets which offer derivatives on several underlying assets, correlation modeling in the context of option pricing theory has been relatively under-investigated in the financial literature.

Correlation swaps appeared in the early 2000's as a means to hedge the parametric risk exposure of exotic desks to changes in correlation. Exotic derivatives indeed frequently involve multiple assets, and their valuation requires a correlation matrix for input. Unlike volatility, whose implied levels have become observable due to the development of listed option markets, implied correlation coefficients are unobservable, which makes the pricing of correlation swaps a perfect example of 'chicken-egg problem.'

In this report, we show how a correlation swap on an equity index can be viewed as a simple derivative on two types of tradable variance, and derive a closed-form formula for its arbitrage price relying upon dynamic trading of these instruments. For this purpose, we start by proposing a 'toy model' for tradable variance in Section 1 which we apply for pricing single volatility derivatives. In Section 2, we introduce a proxy for the payoff of correlation swaps that has the property of involving only tradable variance payoffs, and we extend the toy model for variance to derive the theoretical price of a correlation swap.

1. A toy model for tradable variance

Our purpose is to introduce a simplified model which can be used to price derivatives on realized volatility. We depart from the traditional stochastic volatility models such as Heston (1993) by modeling directly the fair price of a variance swap with the same maturity as the derivative. Here, the underlying tradable asset is the variance swap itself which, at any point in time, is a linear mixture between past realized variance and future implied variance.

This approach lacks the sophistication of other methods and does not address the issue of possible arbitrage with other derivatives instruments. But its simplicity allows us to find closed-form formulas based on a reduced number of intuitive parameters, so that everyone can form an opinion on the rationality of our results.

The Model

In this section we limit our considerations to a market with two tradable assets: variance and cash. We follow in part the guidelines by Duanmu (2004) to introduce a simplified, 'toy model' for the variance asset which is a straightforward modification of the Black-Scholes model for asset prices. We make the usual economic assumptions of constant interest rate r, absence of arbitrage, infinite liquidity, unlimited short-selling, absence of transaction costs, and continuous flow of information. We have the usual set up of a probability space (Ω, F, P) with Brownian filtration (F_t) and an equivalent risk-neutral pricing measure Q.

We further assume that only the variance swap is tradable, but not the Asset itself[1]. Let $v_t(0, T)$ be the price at time t of the floating leg of a variance swap for the period $[0, T]$ where T denotes the maturity or settlement date of the swap. From now on we use the reduced notation v_t and we use the terms 'variance' and 'variance swap' interchangeably.

We specify the dynamics of (v_t) through the following diffusion equation under the risk-neutral measure Q:

$$dv_t = rv_t dt + 2\omega \frac{T-t}{T} v_t dW_t$$

where r and ω are model parameters corresponding to the short-term interest rate and the volatility of volatility, and (W_t) is a standard Brownian motion under Q.

Hence v_0 is the price at inception of the variance swap which can be observed on the market or calculated using the replicating portfolio of puts and calls described in e.g. Demeterfi-Derman (1999); and v_T is the price of the same variance swap at maturity which coincides with the realized variance for the period $[0, T]$.

Our toy model for variance is thus a log-normal diffusion whose volatility parameter linearly collapses to zero between the start date and the maturity date. Note that by Ito-Doeblin this is equivalent to assume that volatility follows a log-normal diffusion with a time-dependent volatility parameter $\omega \frac{T-t}{T}$.

Comparison with stochastic volatility models

We now make the comparison with standard stochastic volatility models of the instantaneous asset variance (X_t). The usual mean-reverting model is

[1] We make this assumption to avoid modeling the Asset price process itself, and escape the debate on model consistency with vanilla option prices. Clearly this is not a realistic assumption, hence the expression 'toy model.'

$$dX_t = \lambda(\theta - X_t)dt + \xi X_t^a dW_t$$

where λ, θ, ξ, a are constant parameters. In this framework the price at time t of a variance swap over the period $[0,T]$ is given by:

$$v_t = e^{-r(T-t)} \frac{1}{T}\left[\int_0^t X_s ds + E\left[\int_t^T X_s ds \middle| F_t\right]\right]$$
$$= e^{-r(T-t)} \frac{1}{T}\left[\int_0^t X_s ds + \theta(T-t) + \frac{1}{\lambda}\left(1-e^{-\lambda(T-t)}\right)(X_t - \theta)\right]$$

This price is independent of the volatility of volatility specification controlled by the parameter ξ. Since the variance swap price is an affine function of the instantaneous variance, the dynamics of (v_t) are straightforwardly obtained:

$$dv_t = rv_t dt + e^{-r(T-t)} \frac{1}{\lambda T}\left(1-e^{-\lambda(T-t)}\right)\xi X_t^a dW_t$$

We may now use the variance swap price expression to obtain the dynamics of (v_t) in terms of v_t only. When a = 1 this simplifies to:

$$dv_t = rv_t dt + \xi\left(v_t - \frac{1}{T}e^{-r(T-t)}\left[\int_0^t X_s ds + \theta(T-t) - \frac{1}{\lambda}\left(1-e^{-\lambda(T-t)}\right)\theta\right]\right)dW_t$$

and we can see that the volatility factor between brackets converges to zero as we approach maturity.

In contrast to the toy model, the volatility specification of the variance swap in a stochastic volatility model is a power of the instantaneous variance, not the variance swap price. For short maturities the two models are comparable.

Terminal distribution

Using the Ito-Doeblin theorem, we can write the diffusion equation for ln v:

$$d\ln(v_t) = \left[r - 2\omega^2\left(\frac{T-t}{T}\right)^2\right]dt + 2\omega\frac{T-t}{T}dW_t$$

Thus, for all times 0 < t < t' < T, we have:

$$v_{t'} = v_t \exp\left[r(t'-t) - 2\frac{\omega^2}{T^2}\int_t^{t'}(T-s)^2 ds + 2\frac{\omega}{T}\int_t^{t'}(T-s)dW_s\right]$$

Calculating the first integral explicitly we obtain:

$$v_{t'} = v_t \exp\left[r(t'-t) - \frac{2}{3}\omega^2 T\left(\left(\frac{T-t}{T}\right)^3 - \left(\frac{T-t'}{T}\right)^3\right) + 2\frac{\omega}{T}\int_t^{t'}(T-s)dW_s\right]$$

In particular, the terminal variance v_T has the following expression:

$$v_T = v_t \exp\left[r(T-t) - \frac{2}{3}\omega^2 T\left(\frac{T-t}{T}\right)^3 + 2\frac{\omega}{T}\int_t^T (T-s)dW_s \right] \quad (1)$$

Furthermore, the stochastic integral $\int_t^T (T-s)dW_s$ has a normal distribution with zero mean and standard deviation $\frac{(T-t)^{3/2}}{\sqrt{3}}$. Thus, v_T has a conditional lognormal distribution with mean $\ln(v_t) + r(T-t) - \frac{2}{3}\omega^2 T\left(\frac{T-t}{T}\right)^3$ and standard deviation $\frac{2}{\sqrt{3}}\omega\sqrt{T}\left(\frac{T-t}{T}\right)^{3/2}$.

Application: arbitrage pricing of volatility derivatives

As an example of an application of our toy model for variance, we derive the arbitrage price of a European contingent claim on realized volatility v_T at maturity. We denote $f(v_T)$ the payoff and $f_t = f(t, v_t)$ the F-adapted price process of such contingent claim.

Following the fundamental theorem of asset pricing, the price of the contingent claim equals the discounted conditional expectation of its payoff:

$$f_t = e^{-r(T-t)} E\left[f(v_T) \middle| F_t \right]$$

We now proceed to derive closed-form formulas for two contingent claims of particular interest:

– A forward contract on realized volatility, whose payoff is the square root of variance: $f(v_T) = \sqrt{v_T}$;

– A call option on realized variance struck at level K, whose payoff is: $f(v_T) = Max(0, v_T - K)$.

Forward contract on realized volatility

Taking the square root of (1) we can write for all $0 < t < T$:

$$f(v_T) = \sqrt{v_t e^{r(T-t)}} \exp\left(-\frac{1}{3}\omega^2 T\left(\frac{T-t}{T}\right)^3 + \frac{\omega}{T}\int_t^T (T-s)dW_s \right)$$

Taking conditional expectations and discounting then yields:

$$f_t = \sqrt{v_t e^{-r(T-t)}} \exp\left(-\frac{1}{3}\omega^2 T\left(\frac{T-t}{T}\right)^3 \right) E\left[\exp\left(\frac{\omega}{T}\int_t^T (T-s)dW_s \right) \middle| F_t \right] \quad (2)$$

But:

$$E\left[\exp\left(\frac{\omega}{T}\int_t^T (T-s)dW_s \right) \right] = \exp\left(\frac{1}{2}\frac{\omega^2}{T^2}\int_t^T (T-s)^2 ds \right) = \exp\left(\frac{1}{6}\omega^2 T\left(\frac{T-t}{T}\right)^3 \right)$$

Substituting this result in (2), we obtain a closed-form formula for the price of the forward contract on volatility:

$$f_t = \sqrt{v_t e^{-r(T-t)}} \exp\left(-\frac{1}{6}\omega^2 T\left(\frac{T-t}{T}\right)^3\right)$$

In particular:

$$f_0 = \sqrt{v_0} \exp\left(-\frac{1}{2}rT - \frac{1}{6}\omega^2 T\right)$$

A corollary is that the convexity adjustment *c* between the fair strikes of newly issued variance and volatility swaps can be expressed as a function of volatility of volatility:

$$c = \sqrt{v_0 e^{rT}} - f_0 e^{rT} = \sqrt{v_0 e^{rT}}\left[1 - \exp\left(-\frac{1}{6}\omega^2 T\right)\right]$$

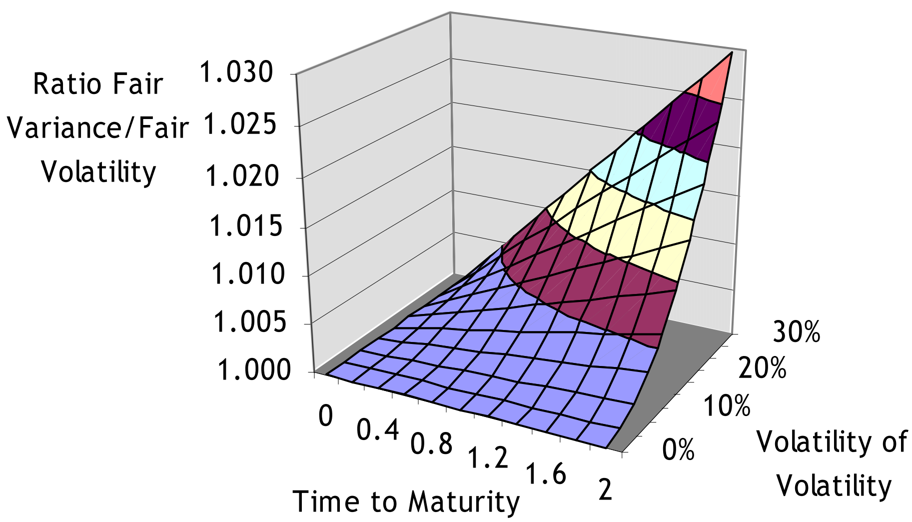
Which gives the rule of thumb:

$$c \approx \frac{1}{6}\sqrt{v_0 e^{rT}}\,\omega^2 T$$

This result has some resemblance with Duanmu's who finds $c = \sqrt{v_0 e^{rT}}\left[1 - \exp\left(-\frac{1}{4}\xi^2\right)\right]$ with a time-dependent volatility of volatility $\omega(T-t) = \frac{\xi}{\sqrt{T-t}}$. However, we believe our result is more consistent with the intuition that the longer the maturity, the higher the convexity effect. Exhibit 1.1 below shows how the convexity adjustment behaves as a function of volatility of volatility and maturity.

Exhibit 1.1

Convexity Adjustment between Variance and Volatility Swaps



Another point of interest is the corresponding dynamic hedging strategy for replicating volatility swaps using variance swaps. Following our approach, the quantity δ_t of variance to hold at a given point in time *t* would be:

$$\delta_t = \frac{\partial f}{\partial v} = \frac{1}{2\sqrt{v_t e^{r(T-t)}}} \exp\left(-\frac{1}{6}\omega^2 T \left(\frac{T-t}{T} \right)^3 \right)$$

It is worth noting that at time *t* = 0 this delta is equal to:

$$\delta_0 = \frac{1}{2\sqrt{v_0 e^{rT}}} \exp\left(-\frac{1}{6}\omega^2 T \right)$$

which is in line, modulus the convexity adjustment, with the market practice of calculating the notional of a newly issued variance swap according to the formula:

$$Variance\ Swap\ Notional = \frac{Vega\ Notional}{2 \times Variance\ Swap\ Strike}$$

Call option on realized variance

Because v_T has a lognormal distribution, the closed-form formula for a call on realized variance struck at level *K* is identical to the Black-Scholes formula for a call on a zero-dividend paying stock with a constant volatility parameter[2] $\sigma = \frac{2}{\sqrt{3}}\omega \frac{T-t}{T}$. Substitution yields:

$$f_t = v_t N(d_1) - K e^{-r(T-t)} N(d_2)$$

where *N* is the cumulative standard normal distribution and:

$$d_1 = \frac{\ln \sqrt{\frac{v_t e^{r(T-t)}}{K}} + \frac{1}{3}\omega^2 T \left(\frac{T-t}{T} \right)^3}{\frac{1}{\sqrt{3}}\omega \sqrt{T} \left(\frac{T-t}{T} \right)^{3/2}},\ d_2 = d_1 - \frac{2}{\sqrt{3}}\omega \sqrt{T} \left(\frac{T-t}{T} \right)^{3/2}.$$

Note that the quantity $\sqrt{\frac{v_t e^{r(T-t)}}{K}}$ corresponds to the ratio of implied variance to the option's strike expressed in volatility points.

Exhibits 1.2 to 1.4 below show how the arbitrage price of a 3-year call on variance struck at 20^2 compares to the original Black-Scholes call formula at *t* = 0, 1 and 2 years. To generate these graphs we assumed a volatility of volatility of 20% and a 0% interest rate. Option prices are expressed in percentage of the strike. We can see that the call on variance is worth more than Black-Scholes at *t* = 0, and less at *t* = 1 and *t* = 2, which indicates a higher time decay or theta.

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[2] By 'constant volatility parameter' we mean that at time t the call on realized variance has the same price as a call on an asset S whose price follows the diffusion $dS_\tau = rS_\tau d\tau + \sigma S_\tau dW_\tau$ where $\tau \in [t, \infty)$ is the time dimension of the diffusion and σ does not depend on τ.

Exhibit 1.2

Call on Realized Variance: Toy Model versus Black-Scholes at t = 0

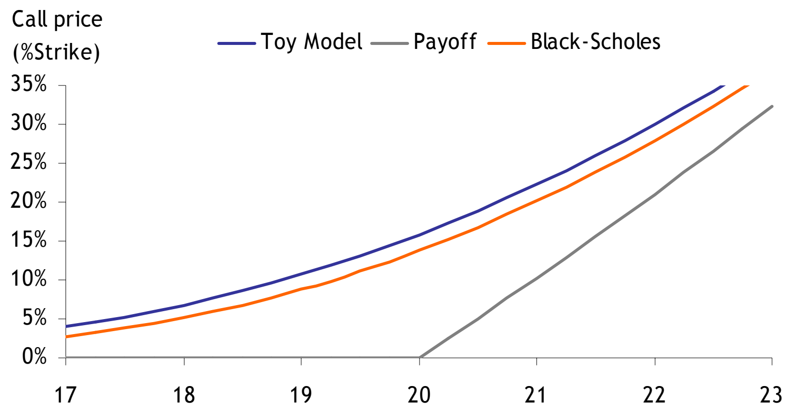


Exhibit 1.3

Call on Realized Variance: Toy Model versus Black-Scholes at t = 1

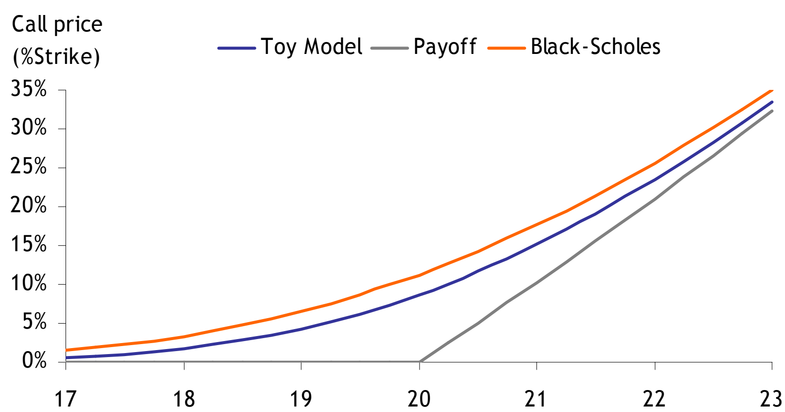
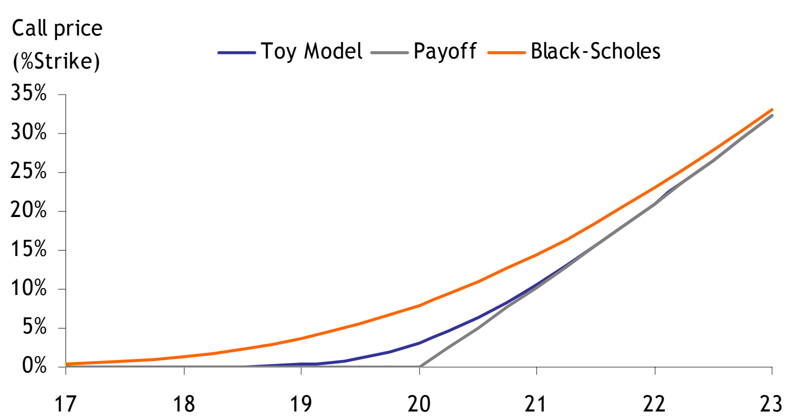


Exhibit 1.4

Call on Realized Variance: Toy Model versus Black-Scholes at t = 2



2. Pricing and hedging of quasi-correlation claims

Correlation Swaps

A correlation swap is a derivative instrument on a basket of *n* Stocks whose payoff is given as:

$$c_T = \frac{\sum_{1 \le i < j \le n} w_i w_j \rho_{i,j}}{\sum_{1 \le i < j \le n} w_i w_j} - K$$

where *w* is a vector of arbitrary non-negative weights summing to 1, ρ a positive-definite matrix of pair-wise correlation coefficients, and *K* a scalar called strike.

In practice the correlation coefficients are calculated using the canonical statistical formula on the time series of the Stocks' daily log-returns. The first term in the formula corresponds to the weighted average of the correlation matrix, excluding the diagonal of 1's. We call this quantity the realized average correlation between the *n* Stocks for the period [0, *T*].

Implied index correlation

In the case of equity indices, an implied average correlation measure can be backed out from implied volatilities:

$$\overline{\rho}_{Implied} = \frac{\sigma_{Index}^2 - \sum_{i=1}^{n} w_i^2 \sigma_i^2}{2 \sum_{1 \le i < j \le n} (w_i \sigma_i)(w_j \sigma_j)}$$

where *n* is the number of Stocks in the index, σ_{Index} is the implied volatility of the index, σ is the vector of implied volatilities and *w* is the vector of index weights.

This measure is justified by the well-known relationship between the variance of a portfolio and the covariance of its components, which is the founding block of portfolio theory (Markovitz 1952):

$$\sigma_{Portfolio}^2 = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + 2 \sum_{1 \le i < j \le n} w_i w_j \sigma_i \sigma_j \rho_{i,j}$$

There are, however, some minor differences between an equity index and a portfolio of stocks. In a portfolio weights are fixed, whereas in an index they vary with stock prices. Additionally the formula above is only exact for standard returns ($\frac{\Delta P}{P}$), not log-returns. In normal market conditions and over reasonable observation periods, these differences can be ignored.

Implied correlation and 'fair' correlation

Intuitively, one would expect the 'fair value' of a correlation swap on an equity index to be related to the index implied correlation. However, in the absence of a replication strategy, the concept of 'fair value' is quite sloppy. This is complicated by the existence of implied volatility surfaces that translate into implied correlation surfaces: there is not a single measure of implied correlation.

Later on we establish the formal existence of a quasi-replication strategy for equity index correlation swaps relying upon dynamic trading of variance on the index and its components, and show that the 'fair value' of a correlation swap is roughly equal to a particular measure of implied correlation, after discounting. This dynamic replication strategy is more easily exposed using the rules of thumb which we introduce below.

Correlation proxy

In 2004 several papers (Bossu-Gu, Tierens-Anadu, Statman-Scheid) have investigated the relationship between portfolio volatility and average correlation. The conclusion which can be drawn is that for a sufficient number of Stocks and in normal conditions[3] we have the rule of thumb:

$$\overline{\rho} \approx \left(\frac{\sigma_{Index}}{\sum_{i=1}^{n} w_i \sigma_i} \right)^2$$

where $\overline{\rho}$ denotes either realized or implied average correlation, σ a vector of either realized or implied volatilities, and *w* a vector of components' weights in the index.

In essence, average correlation is thus the squared ratio of index volatility to the average volatility of its components. We push this paradigm one level further by noticing that this proxy measure is conceptually close to the ratio of index variance to the average variance of its components:

$$\left(\frac{\sigma_{Index}}{\sum_{i=1}^{n} w_i \sigma_i} \right)^2 \approx \frac{\sigma_{Index}^2}{\sum_{i=1}^{n} w_i \sigma_i^2}$$

We call the quantity on the left-hand side the volatility-based correlation proxy and that on the right-hand side the variance-based proxy. In practice those two proxy measures typically differ by a few correlation points for the major equity indices, both for implied and historical data. It should also be noted that the variance-based proxy is always lower than or equal to the volatility-based one[4].

Our motivation for introducing the variance-based proxy should be clear: in this form, average index correlation becomes the ratio of two tradable types of variances: index variance and average components' variance. In fact these variances are frequently traded one against the other in the so-called variance dispersion trades, with the objective of taking advantage of the gap between implied correlation and realized correlation, as illustrated in Exhibit 2.1 on the Dow Jones EuroStoxx 50 index.

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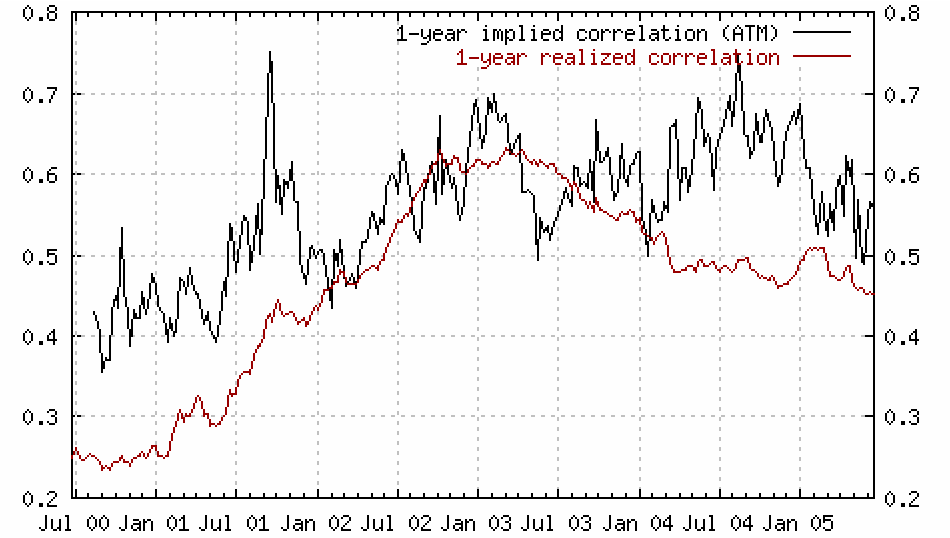
[3] Well-behaved weights and volatilities, actual correlation above 0.15.

[4] This property is a straightforward consequence of Jensen's inequality: $\left(\Sigma_i w_i \sigma_i\right)^2 \le \Sigma_i w_i \sigma_i^2$.

Exhibit 2.1

1-year Implied and Realized Correlation of the DJ EuroStoxx 50 index



Source: JPMorgan

Quasi-correlation claims

We call a quasi-correlation claim a variance derivative whose payoff is:

$$c_T = \frac{a_T}{b_T}$$

where a_T denotes index realized variance and b_T the average components' realized variance, defined as follows:

$$a_T = \frac{1}{T}[\ln S]_T$$

$$b_T = \frac{1}{T}\sum_{i=1}^{n} w_i [\ln S^i]_T$$

with S denoting the price process of the index, (S^1 , …, S^n) the vector of price processes of the components, and [.] the quadratic variation.

Arbitrage Pricing

We now extend our toy model to find the arbitrage price of a quasi-correlation claim. We consider a market of two tradable variance assets a and b and cash, and we make the same economic assumptions as in Section 1.

We specify the following dynamics for the F-adapted price processes a and b under a risk-neutral measure Q:

$$da_t = ra_t dt + 2\omega_a \frac{T-t}{T} a_t dW_t$$

$$db_t = rb_t dt + 2\omega_b \frac{T-t}{T} b_t \left(\chi dW_t + \sqrt{1-\chi^2}\, dZ_t\right)$$

where r is the short-term interest rate, ω's are volatility of volatility parameters for a and b, χ is the instant correlation parameter between a and b, and (W_t), (Z_t) are two independent Brownian motions under Q.

Denoting (c_t) the price process for the quasi-correlation claim, and applying the Ito-Doeblin theorem on $\ln(a/b)$, we find:

$$d\ln\frac{a_t}{b_t} = 2\left(\omega_b^2 - \omega_a^2\right)\left(\frac{T-t}{T}\right)^2 dt + 2(\omega_a - \omega_b\chi)\frac{T-t}{T} dW_t - 2\omega_b\sqrt{1-\chi^2}\,\frac{T-t}{T} dZ_t$$

Whence for all $0 < t < T$:

$$c_T = \frac{a_T}{b_T} = \frac{a_t}{b_t}\exp\left[\frac{2}{3}\left(\omega_b^2 - \omega_a^2\right)T\left(\frac{T-t}{T}\right)^3 + 2\frac{\omega_a - \omega_b\chi}{T}\int_t^T (T-s)dW_s - 2\frac{\omega_b\sqrt{1-\chi^2}}{T}\int_t^T (T-s)dZ_s\right]$$

Taking conditional expectations under Q and discounting yields:

$$c_t = \frac{a_t}{b_t}\exp\left[-r(T-t) + \frac{2}{3}\left[\omega_b^2 - \omega_a^2 + (\omega_a - \omega_b\chi)^2 + \omega_b^2(1-\chi^2)\right]T\left(\frac{T-t}{T}\right)^3\right]$$

Expanding the squares and simplifying terms, we obtain:

$$c_t = \frac{a_t}{b_t}\exp\left[-r(T-t) + \frac{4}{3}\left(\omega_b^2 - \omega_a\omega_b\chi\right)T\left(\frac{T-t}{T}\right)^3\right]$$

In particular, at time $t = 0$, we have:

$$c_0 = \frac{a_0}{b_0}\exp\left[-rT + \frac{4}{3}\left(\omega_b^2 - \omega_a\omega_b\chi\right)T\right]$$

Here, it is worth noting that if the volatility of volatility parameters are of the same order and the correlation of variances is high, we have $c_0 \approx \frac{a_0}{b_0}e^{-rT}$, which is nothing else but the discounted variance-based implied correlation proxy.

Exhibits 2.2 to 2.4 below show how the fair strikes of a 1-year quasi-correlation claim compare to the variance-based implied correlation, for various levels of volatility of volatility and variance correlation parameters. We can see that when ω_a and ω_b are close the ratio is close to 1. This suggests that our result is relatively model-independent in the sense that it does not heavily depend on the model parameters.

Exhibit 2.2

Ratio of Fair Quasi-Correlation to Variance-Based Implied Correlation for $\chi = 0.5$

ω_a \ ω_b	0%	5%	10%	15%	20%	25%	30%
0%	1	1.003	1.013	1.030	1.055	1.087	1.127
5%	1	1.001	1.009	1.023	1.045	1.074	1.112
10%	1	0.999	1.004	1.016	1.035	1.062	1.096
15%	1	0.996	0.999	1.009	1.025	1.049	1.081
20%	1	0.994	0.994	1.002	1.016	1.037	1.065
25%	1	0.992	0.990	0.995	1.006	1.025	1.051
30%	1	0.989	0.985	0.988	0.997	1.013	1.036

Exhibit 2.3

Ratio of Fair Quasi-Correlation to Variance-Based Implied Correlation for $\chi = 1$

ω_a \ ω_b	0%	5%	10%	15%	20%	25%	30%
0%	1	1.003	1.013	1.030	1.055	1.087	1.127
5%	1	1	1.007	1.020	1.041	1.069	1.105
10%	1	0.997	1	1.010	1.027	1.051	1.083
15%	1	0.993	0.993	1	1.013	1.034	1.062
20%	1	0.990	0.987	0.990	1	1.017	1.041
25%	1	0.987	0.980	0.980	0.987	1	1.020
30%	1	0.983	0.974	0.970	0.974	0.983	1

Exhibit 2.4

Ratio of Fair Quasi-Correlation to Variance-Based Implied Correlation for $\chi = 0$

ω_a \ ω_b	0%	5%	10%	15%	20%	25%	30%
0%	1	1.003	1.013	1.030	1.055	1.087	1.127
5%	1	1.002	1.010	1.025	1.048	1.078	1.116
10%	1	1.000	1.007	1.020	1.041	1.069	1.105
15%	1	0.998	1.003	1.015	1.034	1.060	1.094
20%	1	0.997	1.000	1.010	1.027	1.051	1.083
25%	1	0.995	0.997	1.005	1.020	1.043	1.073
30%	1	0.993	0.993	1.000	1.013	1.034	1.062

Dynamic Hedging Strategy

We now examine in further detail the hedging strategy for quasi-correlation claims. The hedging coefficients or deltas for the two variance assets are given as:

$$\delta_t^a = \frac{c_t}{a_t}$$

$$\delta_t^b = -\frac{c_t}{b_t}$$

In practice, this means that if we are short a claim we must hold a long position in index variance against a short position in average components' variance, in dynamic quantities. This type of spread trade is known as a variance dispersion. We must emphasize that here the weights between the two legs are not equal — in fact, the ratio of deltas is equal to the fair value of the claim:

$$\left|\frac{\delta_t^b}{\delta_t^a}\right| = \frac{a_t}{b_t} = c_t$$

In particular, at $t = 0$, this ratio is equal to the variance-based implied correlation proxy, and the initial delta-hedge is known as a correlation-weighted variance dispersion trade[5].

Furthermore, the cost of setting up the delta-hedge is nil at all times:

$$\delta_t^a a_t + \delta_t^b b_t = \frac{c_t}{a_t} a_t - \frac{c_t}{b_t} b_t = 0$$

Thus, the hedging strategy is entirely self-funded.

[5] For a detailed analysis of dispersion trading, please refer to our 2005 report *Correlation Vehicles*, JPMorgan European Equity Derivatives Strategy, N. Granger and P. Allen.

Conclusion

Because standard correlation swaps have a payoff approximately equal to that of a quasi-correlation claim minus the strike, it follows that the hedging strategy for the latter is a quasi-replication strategy for the former in the sense that it replicates the payoff modulus the error of the correlation proxy. In other words, correlation swaps on an equity index should trade at a strike close to the variance-based implied correlation proxy. It should be pointed out that at the time of writing, over-the-counter transactions typically take place at a significantly lower strike, which may indicate the existence of dynamic arbitrage opportunities.

The implications are vast from both practical and theoretical standpoints. On the practical side, the identification of a quasi-replication strategy is a crucial step for the development of the correlation swap market. On the theoretical side, we see at least three research areas which should be affected by our results: the pricing and hedging of exotic derivatives on multiple equity assets (in particular the long-debated issue of correlation skew), the stochastic modeling of volatility and correlation, and the pricing and hedging of options on realized correlation as a branch of the pricing theory of derivatives on realized variance.

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